The Relationship between Buffon's Needle Problem and π

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Abstract

The purpose of the experiment was to verify whether Buffon's Needle Problem could be used to approximate π. If π could be approximated from the problem, then it demonstrates how π is not derived from just elliptical objects, but also from other areas in physics and mathematics. The experiment hypothesis stated "the smaller the difference between *d* (distance between lines) and *l* (needle length), then the nearer the probability of touching needles will be to 2/π". To verify whether Buffon's Needle Problem could approximate π, 3.81 centimeter needles were dropped on boards with equidistant parallel lines running across them. The distance between lines: ¼, ½, 1 - the control, 2, and 4 (relative to needle length), were the levels of the independent variable, and the number of needles on lines was the dependent variable. The collected data revealed that the closer *d* and *l* were, the nearer the probability of touching needles were to 2/π, the probability of the control (where *d* is equal to *l*) being 0.66, comparatively much closer to the probability of 2/π than the other probabilities. Multiple paired samples t-tests with df = 49 were conducted at the significance level of 0.05 between each level of the independent variable. All the t-tests had p-values of below 0.00001, which demonstrated the results were significant, so a conclusion was drawn from the data. The conclusion supported the experiment hypothesis, and verified that Buffon's Needle Problem could be used to approximate π as well.

Keywords: needles on lines, Buffon, 2/pi

Introduction

π is the value most commonly found through the ratio between a circle's circumference and its diameter. As it is an irrational number, (does not repeat and goes on infinitely) most people estimate it to be numbers such as 22/7, 3.14, or 3.14159. Throughout history, Babylonians, Egyptians, Greek, Arab, Chinese, and Indian mathematicians have all attempted to estimate π through a number of methods, though recently calculations have become exponentially more precise with the help of computers, with the best estimations at 31 trillion digits. Attempts to calculate π have been made so many times throughout history as it is integral to calculating lengths of curves and areas of ellipse and sectors, and is also used to calculate the volume of some solids. Because of π's involvement in calculating the dimensions of elliptical objects, other ways to approximate π have been invented that few people know about, such as with Buffon's needle problem.

Georges-Louis Leclerc, count de Buffon, was a French naturalist who was best known for his extensive work in the field of natural history, especially on his volumes, Natural History, General and Particular, with his famous name Buffon originating from his mother's estate. As a young man, though his father wanted him to study law at the College of Godrans in Dijon, he transferred to Angers, where he studied mathematics, botany, and medicine. However, he had to leave after a duel and visited Rome and England with his friend, the latter of which being the location where he became a member of the Royal Society. However, he returned to France after the death of his mother, where he researched the calculus of probability, though his main focus was plant physiology. He was later appointed keeper of France's royal gardens.

It was in these gardens where he was told to catalog the French royal collections of natural history, though he would instead use the opportunity to make an account for all of nature. This became the work he was most known for, " Histoire naturelle, générale et particulière", which attempted to display all existing knowledge of anthropology, natural history, and geology, in a single publication. However, he never finished it, only completing 36 of the intended 50 volumes before his death. His wife died in 1769, leaving him with a son of only 5 years of age. However, the boy would waste his opportunities to study, spending money irresponsibly, and eventually would be executed via guillotine during the French Revolution. Buffon's health declined in 1785, and in 1788, knowing his end was nigh, he would return to Paris, where he stayed in his room, accompanied by his good friend Mme Necker until he passed away.

Relating to the needle problem, Eric Weisstein (2003) states that Buffon first pondered about the needle problem in 1733, later providing a solution for it in 1777. Alexander Bogomolny (1999) says that he first considered the probability of a circle landing within a square before thinking of the needle problem: the probability of an l long shaped needle landing on one of many d long equally spaced parallel lines. As for the experiment, because it is helpful to view anything from multiple perspectives, and considering the relative obscurity of other ways to find π, this experiment shall be conducted in order to both prove there are other ways to find π besides the standard of the ratio between a circle's circumference and its diameter and to show that Buffon's needle problem can be used to estimate π as well.

Buffon's needle problem is hypothesized to be one such way and forms the basis for the question: "What is the probability of an l long shaped needle landing on one of many d long equally spaced parallel lines?" As needles will be used, there are several different possible orientations for it. Alexander Bogomolny (1999) states that the needle's range of orientations can run from $\theta=0°$ to $\theta=180°$, and all orientations are equally likely. This means that the orientation of the needle should not be restricted to one orientation as over time the needle orientations will resemble the average. The hypothesis is that "the smaller the difference between d and l, then the nearer the probability of touching needles will be to $2/\pi$", and a mathematical explanation of this will be attempted. As for the experimentation itself, the length of the distance between the parallel lines will be the independent variable (with 4 levels, measured in relation to the needles: ¼, ½, 1, 2, and 4, with the length of the needle being equal to the distance between lines, or 1, as the control) , and the probability of touching needles compared to non-touching needles will be the dependent variable.

Methodology

This experiment was conducted upon the carpeted basement floor of a 2-story suburban house. First, 20 identical 3.81 centimeter long steel needles were bought and put inside a small container for safekeeping. A funnel apparatus was then constructed for more consistent results, which first used two 30.48-centimeter long cubic cardboard boxes situated 66.04 centimeters apart from one another in a straight line so as to prevent any needles from hitting the boxes and to make sure the needles did not bounce off the cardboard plane that they landed on. Two 96.52 centimeter-long, 9.525 centimeter-tall, 0.5 centimeter-wide wooden planks were then placed on top of the opposite sides of the two cubes so they formed two bridges parallel to the box's sides with 15.24 centimeters of wood laying on each end of either box. A styrofoam square cuboid that is 30.48 centimeters long by 30.48 centimeters wide by 1.27 centimeters in height with a hole in the middle was placed directly in the middle between the boxes on the two wood planks. The cuboid was styrofoam as it is a material that can easily be punched through but also has enough structural integrity to support itself. A 15.24-centimeter long circular plastic funnel was then inserted into the styrofoam hole which was secure enough to prevent the funnel from falling out. The funnel was plastic as it was the easiest to acquire. Finally, one of 5 cardboard planes was selected on which to drop the needles on, with parallel lines running across it. Cardboard was used because of its rigid surface.

These lines on the cardboard were equally spaced apart to maintain consistency, and depending on the plane, were 0.9525 centimeters apart, 1.9050 centimeters apart, 3.81 centimeters apart, 7.62 centimeters apart, and 15.24 centimeters apart (notably, the distances were, in relation to the length of the needle, ¼, ½, 1, 2, and 4 times as long) as the different levels of the independent variable. Once 250 trials of dropping 20 needles from the funnel onto the surface of the cardboard plane were run, the plane was then switched to one with a different distance between the lines until all the planes were tested and 50 trials were run, with 250 needles dropped in total. The results of each of these 50 trials were recorded on a Google Sheets (used for its wide variety of tools) and were completed over the course of ~4 weeks (~28 days). The ratio between the needles that touch the lines and the non-touching needles for each level of the independent variable was measured and the hypothesis 'the smaller the difference between *d* and *l* (*d* being the distance between the parallel lines and *l* being the length of the needles) , then the nearer the probability of touching needles compared to non-touching needles will approach 2/π' was either supported or rejected.

Data and Results

Table 1:

	All Needles On and Off Lines (Raw Data Table)									
	0.9525 cm. space		1.9050 cm. space		3.81 cm. space		3 cm. space		6 cm. space	
Trial Number	On line	Off line	On line	Off line	On line	Off line	On line	Off line	On line	Off line
1	20	0	18	2	12	8	9	11	4	16
2	19	1	19	1	18	2	10	10	2	18
3	18	2	17	3	12	8	9	11	3	17
4	19	1	19	1	11	9	5	15	3	17
5	20	0	17	3	16	4	5	15	2	18
6	19	1	19	1	13	7	5	15	3	17
7	19	1	16	4	16	4	7	13	2	18
8	20	0	17	3	11	9	6	14	3	17
9	18	2	18	2	12	8	7	13	1	19
10	19	1	19	1	13	7	5	15	4	16

Figure 1:

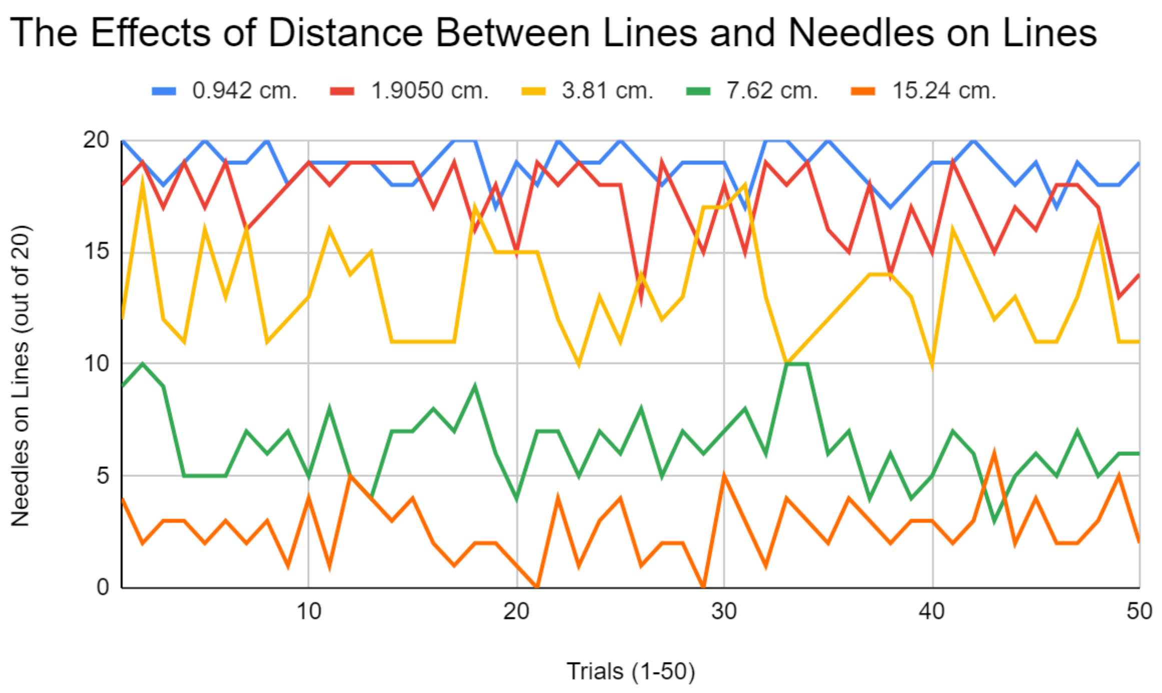

Figure 1 shows the number of needles that landed on the equally spaced lines for each of the levels of the independent variable. The hypothesis of the project was: "the smaller the difference between *d* and *l*, then the nearer the probability of touching needles will be to 2/π", derived from Buffon's understanding of the problem, with *d* being the distance between lines and *l* being the length of the needles. Thus, the independent variable was the distance between lines while the dependent variable would be the number of needles that landed on the lines, shown, respectively, as the x and y-values of Figure 1.

Table 2:

Descriptive Statistics Table					
Level of IV	0.9525 cm	1.9050 cm	3.81 cm	7.62 cm	15.24 cm
Mean	18.84	17.24	13.2	6.4	2.66
Median	19	18	13	6	3
Probability	0.942	0.862	0.66	0.32	0.133
Variance	0.75	3.00	4.98	2.69	1.74
Standard Deviation	0.87	1.73	2.23	1.64	1.32
1 SD (68%)	17.97-19.71	15.51-18.97	10.97-15.43	4.76-8.04	1.34-3.98
2 SD (95%)	17.1-20.58	13.78-20.7	8.74-17.66	3.12-9.68	0.02-5.3
3 SD (98%)	16.23-21.45	12.05-22.43	6.51-19.89	1.48-11.32	-1.3/6.62
Number	50	50	50	50	50
Results of t test	t=-16.024, p<.00001	t=-9.733, p<.00001		t=18.200, p<.00001	t=26.203, p<.00001
(df=49, a=0.05)	significant	significant		significant	significant

Table 1 displays a portion of the raw data table, while Table 2 displays the descriptive statistics table. As shown in the means section, the means gradually started decreasing as the distance between the lines grew, as did the medians, from 18.84 to 2.66. The control, 3.81 centimeters (same as needle length), had the highest variance at 4.98, while 0.9525 centimeters had the lowest variance of 0.75. Table 2 also displays the results of the t-tests. The null hypothesis was that the number of needles landing on lines would be the same for all

5 different distances between lines and paired samples t-tests with df=49 were conducted at the significance level of 0.05 to test it. The comparison between the 3.81 cm. control distance (mean = 13.2, SD = 2.23) and the 0.9525 cm. distance (mean = 18.84, SD = 0.87), showed a significant difference in the amount of needles landing on the lines, $t(49)=-16.024$, $p<.00001$. In this case, the null hypothesis was rejected, and the alternate hypothesis was not rejected. Likewise, the comparison between the 3.81 cm. control distance and the 1.9050 cm. distance (mean = 17.24, SD = 1.73) also showed a significant difference in the number of needles landing on the lines, $t(49)=-9.733$, $p<.00001$.

As such, the null hypothesis was again rejected and the alternate hypothesis was not rejected. For the comparison between the 3.81 cm. control distance and the 7.62 cm. distance (mean = 6.4, SD =1.64), there was also a significant difference in the number of needles landing on the lines, $t(49)=18.200$, $p<.00001$. Therefore, the null hypothesis was still rejected in this case, and the alternate hypothesis was not rejected. Finally, the comparison between the 3.81 cm. control distance and the 15.24 cm. distance was also revealed to have a significant difference in the amount of needles landing on the lines, $t(49)=26.203$, $p<.00001$. Overall, the data had not immediately rejected the research hypothesis that if "the smaller the difference between d and l, then the nearer the probability of touching needles will be to $2/\pi$" but some more calculations had to be done to prove whether the probability of the needles dropping on the lines from the levels of the independent variables were nearing the value of $2/\pi$ as the value of d became closer to l.

Figure 2:

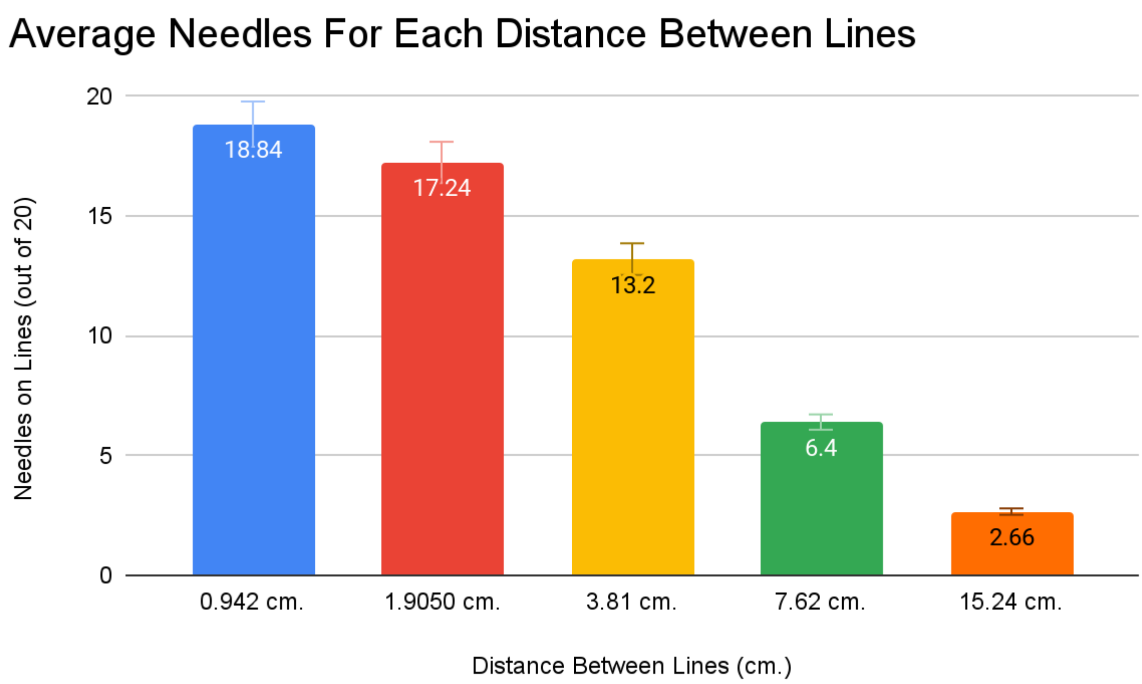

Figure 2 displays the mean number of needles that fell onto lines out of the 5 separate levels of the independent variable, with the differences between the levels all being significant. It was noted from the graph that there was an inverse relationship between the number of needles dropping on lines and the distance between the lines, as indicated by the trend shown in the graph. As for calculating the probability, the number of needles falling onto the lines in each trial was out of 20, and so by dividing these numbers by 20, their average probabilities were able to be deduced, with the control where *d*=*l*, 13.2 divided by 20 (0.66) was noticed to be very close to 2/π(which in the case of this experiment will be rounded to 0.64), and the other distances gradually became less close with 0.942 as the probability for 0.9525 cm, 0.862 as the probability for 1.9050 cm, 0.32 as the probability for 7.62 cm, and 0.133 as the probability for 15.24 cm.

Conclusion

The purpose of this experiment was to verify whether Buffon's Needle Problem could be used to approximate π. If it could be used to approximate π, then it demonstrates how π is not just derived from elliptical objects, but also from various other areas in physics and mathematics. Of course, π's importance is highlighted in its usage for calculating lengths of curves and areas of ellipse and sectors, and is also used to calculate the volume of some solids. That is why, even though calculating closer estimates to π is unnecessary, most people should still remember a few ways to approximate π.

Buffon's Needle Problem thus poses the question "What is the probability of an l long shaped needle landing on one of many d long equally spaced parallel lines?", which was used for the research question. The experiment hypothesis stated "the smaller the difference between d and l, then the nearer the probability of touching needles will be to 2/π", with 2/π being derived from Buffon's understanding of the problem. For the experiment, twenty 3.81 cm. needles were utilized, as well as 5 large, flat sheets of cardboard and a funnel apparatus. The independent variable was the distance between lines (¼, ½, 1, 2 , and 4 were the distances measured in relation to the length of the needles), 1 (3.81 cm.) being the control, and the dependent variable was the number of needles that landed on said lines. To test the hypothesis, 250 trials were run, with 50 trials for each of the 5 levels of the independent variable. These trials consisted of dropping the 20 needles onto one of five planes with equidistant parallel lines, spaced accordingly with the levels of the independent variable.

The collected data revealed that the average probability of needles landing on lines went down as the distance between lines increased. The probabilities are derived by dividing the mean of each level of the IV by 20 (the number of needles dropped in each trial). This was expected, as if there are less lines in a given space, then objects are expected to be less likely to interact with the lines in said space. Nevertheless, several one-tailed t-tests with df = 49 were conducted at the significance level of 0.05 between the datasets from each level of the IV. All the t-tests had p-values below 0.00001, well below the significance level of 0.05, meaning that the differences between levels were statistically significant. The average probability gradually became closer to 2/π as the distance between lines grew closer to the needle length of 3.81 cm, with 94.2% as the probability for 0.9525 cm, 86.2% as the probability for 1.9050 cm, 32% as the probability for 7.62 cm, and 13.3% as the probability for 15.24 cm, with the closest being the control (where the distance between lines is the same as needle length). Thus, the data does support the experiment hypothesis "the smaller the difference between *d* and *l*, then the nearer the probability of touching needles will be to 2/π", as derived.

However, the probability of the needles landing on lines for the control, 13.2/20, was slightly higher than the expected value, 2/π. This was most likely due to random chance, but it should be noted that the lines drawn were somewhat wide, and could have caused the results to err in accuracy. To account for this, needles that landed on the lines were checked to have landed on at least 50% of the line for them to be considered on the line. In addition, some modifications were made to the funnel apparatus which dropped the needles so that less needles would bounce off the cardboard after some data had already been collected, which could have altered the consistency of the results. Lastly, the boards had some imperfections in them that caused the needles to be more likely to fall into these areas. Though, if such was the case, the needles were dropped again until they landed elsewhere. In any case, further extensions of this experiment could be made. First of all, for any further extension, the equipment and procedures should be more polished, with the boards being more smooth with narrower lines, and with the funnel apparatus being adjusted at the beginning of the trials instead. This would ensure that the data being collected was more accurate and consistent. As for extensions, one could discover whether the probability of other ratios between *d* and *l* had any relationship to π or other mathematical constants. This could show whether Buffon's Needle Problem has an inherent connection with π, or perhaps it could be used to approximate other mathematical constants, if applied in a different way. Another possible extension would be to replicate the experiment, but instead of using needles, other shapes such as circular disks could be used instead, with the but with the greatest (in number) dimension of the shape, such as the diameter of the circle replacing the length of the needle as l. For shapes such as circular disks that will always land on a line if l is larger than d, this will force trials to be run where the levels of the independent variable can only reflect distances between lines greater than l. This would give greater insight on the nature of Buffon's Needle Problem and help one determine if changing the object dropped can reveal new relationships between the probability and special values such as mathematical constants and irrational numbers. Perhaps both of these proposed extensions could even be combined, to find whether other shapes and other ratios between d and l have relationships with the aforementioned special values. These ideas are just a few proposals for possible extensions of this experiment.

Appendix: (additional data tables)

https://docs.google.com/spreadsheets/d/1UpXMAxsw5SnLpHzK1E5B4hr461XXXCqkrLDWheBqO9k/edit?usp=sharing

References

Bogomolny, A. (1999, May 22). *Buffon's Needle Problem*. Interactive Mathematics Miscellany and Puzzles. Retrieved September 18, 2021, from http://www.cut-the-knot.org/fta/Buffon/buffon9.shtml.

Britannica, T. Editors of Encyclopaedia (2021, September 17). pi. Encyclopedia Britannica. Retrieved October 26, 2021, from https://www.britannica.com/science/pi-mathematics

Piveteau, J. (2021, September 3). Georges-Louis Leclerc, count de Buffon. Encyclopedia Britannica.

Retrieved October 26, 2021, from

https://www.britannica.com/biography/Georges-Louis-Leclerc-comte-de-Buffon

Posamentier, A., & Lehmann, I. (2004). *Pi: A biography of the world's most mysterious number*. Prometheus Books.

Weisstein, E. W. (2003, July 7). *Buffon's Needle Problem*. from Wolfram MathWorld. Retrieved September 17, 2021, from https://mathworld.wolfram.com/BuffonsNeedleProblem.html.